JOURNAL OF APPROXIMATION THEORY **58**, 242–243 (1989)

Note

A Note on Convergence of Quadratic Interpolatory Splines

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Communicated by Oved Shisha

Received June 27, 1986

Marsden [1, 2] has obtained some error estimates concerning quadratic splines interpolating given functional values at the midpoints between the knots. In this note we point out that it is possible to improve three such estimates. Adopting the definitions and notation used in Marsden [1], we shall prove the following

THEOREM. *Let y and y' be continuous and periodic functions. Then* (i) $\|e_i'\| \leqslant (5/2)\, w(y', p/2)$; (ii) $\|e'\| \leqslant 4w(y', p/2)$; (iii) $\|e\| \leqslant 2pw(y', p/2)$.

Proof of the Theorem. A little calculation shows that the following equalities hold.

$$\begin{aligned} p_i^2 s(x) = {} & p_i(x - x_{i-1})\, s_i + p_i(x_i - x)\, s_{i-1} \\ & + 4(x - x_{i-1})(x_i - x)[\, y_{i-(1/2)} - \tfrac{1}{2}(s_i + s_{i-1})], \end{aligned} \tag{1}$$

$$p_{i+1} s_{i-1} + 3(p_i + p_{i+1})\, s_i + p_i s_{i+1} = 4(p_{i+1}\, y_{i-(1/2)} + p_i\, y_{i+(1/2)}), \tag{2}$$

and

$$p_i \lambda_{i-1} + 3(p_i + p_{i+1})\, \lambda_i + p_{i+1} \lambda_{i+1} = 8(y_{i+(1/2)} - y_{i-(1/2)}), \tag{3}$$

where $s(x_i) = s_i$, $s'(x_i) = \lambda_i$, $x_i - x_{i-1} = p_i$ and p-maximum of p_i over all i. Replacing λ_i by $e_i' + y_i'$ in (3) we see after an appropriate application of Taylor's Theorem that the right-hand side of (3) becomes,

$$\begin{aligned} & p_{i+1}(y'(\theta_2) - y'_{i+1}) + 3p_{i+1}(y'(\theta_2) - y_i') \\ & \quad + 3p_i(y'(\theta_1) - y_i') + p_i(y'(\theta_1) - y'_{i-1}), \end{aligned}$$

0021-9045/89 \$3.00

where $\theta_1 \in (x_{i-(1/2)}, x_i)$ and $\theta_2 \in (x_i, x_{i+(1/2)})$. This combined with (3) leads to (i).

For the next two results we assume without loss of generality that, in (x_{i-1}, x_i), the maximum of $e'(x)$ (respectively, $e(x)$) is assumed at a point $> x_{i-(1/2)}$. We see that

$$p_i e'(x) = (x_i - x)\, e'_{i-1} + (x - x_{i-1})\, e'_i + (x_i - x)(y'_{i-1} - y'_{i-(1/2)}) + (x - x_{i-1})(y'_i - y'(x)) - (x_i - x)(y'(x) - y'_{i-(1/2)}).$$

Thus

$$|e'(x)| \leqslant \|e'_i\| + [1 + (x_i - x)/p_i]\, w(y', p/2)$$

which directly gives (ii).

Setting $x - x_{i-1} = a$ and $x_i - x = b$, we see from (1), after applying Taylor's Theorem, that

$$p_i^2 e(x) = (a-b)[a e_i - b e_{i-1} + b\{a(y'(\theta_3) - y'_{i-(1/2)}) + \tfrac{1}{2}(a+b)(y'(\theta_1) - y'_{i-(1/2)}) + \tfrac{1}{2}(3a+b)(y'_{i-(1/2)} - y'(\theta_2))\}],$$

where $\theta_1 \in (x_{i-1}, x_{i-(1/2)})$ and $\theta_2,\ \theta_3 \in (x_{i-(1/2)}, x_i)$. Thus

$$|e(x)| \leqslant \frac{(a-b)}{p_i}\left[\|e_i\| + \frac{b(a+b)+2ab}{p_i}\, w(y', p/2)\right]$$

which leads to (iii) when we appeal to the estimate $\|e_i\| \leqslant p w(y', p/2)$ as obtained by Marsden [1, Theorem 2.2].

Acknowledgment

The author thanks Prof. H. P. Dikshit.

References

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